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INCOMPRESSIBILITY ESTIMATES FOR THE LAUGHLIN PHASE

NICOLAS ROUGERIE AND JAKOB YNGVASON

Abstract. This paper has its motivation in the study of the Fractional Quantum Hall Effect. We consider 2D quantum particles submitted to a strong perpendicular magnetic field, reducing admissible wave functions to those of the Lowest Landau Level. When repulsive interactions are strong enough in this model, highly correlated states emerge, built on Laughlin's famous wave function. We investigate a model for the response of such strongly correlated ground states to variations of an external potential. This leads to a family of variational problems of a new type. Our main results are rigorous energy estimates demonstrating a strong rigidity of the response of strongly correlated states to the external potential. In particular we obtain estimates indicating that there is a universal bound on the maximum local density of these states in the limit of large particle number. We refer to these as incompressibility estimates.

Contents

1. Introduction

The introduction by Laughlin of his famous wave-function [Lau, Lau2] forms the starting point of our current theoretical understanding of the fractional quantum Hall effect (FQHE) [BF, Goe, Gir, STG], one the most intriguing phenomena of condensed matter physics. The situation of interest is that of interacting particles confined in two space

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dimensions, submitted to a perpendicular magnetic field of constant strength. The Hamiltonian of the full system is given by

$$H_N = \sum_{j=1}^{N} \Big(-\left(\nabla_j - \mathrm{i}\mathbf{A}(\mathbf{x}_j)\right)^2 + V(\mathbf{x}_j) \Big) + \lambda \sum_{1\leq i<j\leq N} w(\mathbf{x}_i - \mathbf{x}_j) \tag{1.1}$$

where $\mathbf{A}$ is the vector potential of the applied magnetic field of strength B, given by

$$\mathbf{A}(\mathbf{x}) = \tfrac{B}{2}(-y, x)$$

in the symmetric gauge. The pair interaction potential is denoted w, the coupling constant λ. Here we think of repulsive interactions, $w \geq 0, \lambda \geq 0$. Units are chosen so that Planck's constant $\hbar$, the velocity of light and the charge are equal to 1 and the mass is equal to 1/2. The potential V can model both trapping of the particles and disorder in the sample. We will study systems made of fermions or bosons, so H_N will act either on $\bigotimes_{\mathrm{as}}^N L^2(\mathbb{R}^2)$ or $\bigotimes_{\mathrm{s}}^N L^2(\mathbb{R}^2)$, the anti-symmetric or symmetric N-fold tensor product of $L^2(\mathbb{R}^2)$.

The quantum Hall effect (integer or fractional) occurs in the regime of very large applied magnetic fields, $B \gg 1$. In this case one can restrict attention to single particle states corresponding to the lowest eigenvalue of $-\left(\nabla - i\mathbf{A}(\mathbf{x})\right)^2$, i.e., the lowest Landau level (LLL)

$$\mathfrak{H} := \Big\{ \psi \in L^2(\mathbb{R}^2) : \; \psi(\mathbf{x}) = f(z) e^{-B|z|^2/4}, \; f \text{ holomorphic } \Big\} \tag{1.2}$$

as the single-particle Hilbert space. Here and in the sequel we identify vectors $\mathbf{x} \in \mathbb{R}^2$ and complex numbers $z = x + iy \in \mathbb{C}$. Henceforth we choose units so that the magnetic field is 2, in order to comply with the conventions for rotating Bose gases as in [RSY1, RSY2] where B corresponds to 2 times the angular velocity.

While the integer quantum Hall effect (IQHE) can be understood in terms of single-particle physics and the Pauli principle[1] only, the FQHE has its origin in the pair interactions between particles[2]. The situation that is best understood is that where the interactions are repulsive enough to force the many-body wave-function to vanish whenever particles come close together. This leads to the following trial states [Lau]

$$\Psi_{\mathrm{Lau}}(z_1, \ldots, z_N) = c_{\mathrm{Lau}} \prod_{1\leq i<j\leq N} (z_i - z_j)^{\ell} e^{-\sum_{j=1}^N |z_j|^2/2} \tag{1.3}$$

where ℓ is an odd number for fermions (the case originally considered by Laughlin is $\ell = 3$) and an even number for bosons ($\ell = 2$ is the most relevant), as imposed by the symmetry. The constant c_{Lau} normalizes the state in $L^2(\mathbb{R}^{2N})$. Much of FQHE physics is based on the strong inter-particle correlations included in Laughlin's wave-function.

More generally, one may restrict the wave function to live in the space

$$\mathrm{Ker}(\mathcal{I}_N) = \left\{ \Psi \in L^2(\mathbb{R}^{2N}) : \; \Psi(z_1, \ldots, z_N) = \Psi_{\mathrm{Lau}}(z_1, \ldots, z_N) F(z_1, \ldots, z_N), \; F \in \mathcal{B}^N \right\} \tag{1.4}$$

[1] The IQHE hence requires the particles to be fermions.

[2] The FQHE can thus in principle also occur in bosonic systems.

where

$$\mathcal{B}^N := \bigotimes_s^N \mathcal{B} = \left\{ F \text{ holomorphic and symmetric} \mid F(z_1, \ldots, z_N) e^{-\sum_{j=1}^N |z_j|^2/2} \in L^2(\mathbb{R}^{2N}) \right\} \tag{1.5}$$

is the N-body bosonic Bargmann space (symmetric here means invariant under exchange of two particles z_i, z_j), with scalar product

$$\langle F, G \rangle_{\mathcal{B}^N} := \left\langle F e^{-\sum_{j=1}^N |z_j|^2/2}, G e^{-\sum_{j=1}^N |z_j|^2/2} \right\rangle_{L^2(\mathbb{R}^{2N})}. \tag{1.6}$$

The restriction of admissible states to $\mathrm{Ker}(\mathcal{I}_N)$ is reminiscent of the use of the Gutzwiller projector in the study of Mott insulators, see [LNW] and references therein.

The choice of the notation $\mathrm{Ker}(\mathcal{I}_N)$ is motivated by the fact that for $\ell = 2$ this is precisely the kernel of the interaction operator

$$\mathcal{I}_N = \sum_{1 \leq i < j \leq N} w(\mathbf{x}_i - \mathbf{x}_j)$$

acting on the N-body bosonic LLL, with w a contact potential $\delta(\mathbf{x})$ [TK, PB, RSY2]. For bosons with short range interactions, the contact interaction can be rigorously proved to emerge in a well defined limit [LS]. We remark that the wave function is an eigenfunction of angular momentum if and only if the correlation factor is a homogeneous polynomial.

The vanishing of Ψ_{Lau} along the diagonals $z_i = z_j$ strongly decreases the interaction energy, and in this paper we will neglect all the eventual residual interaction, an approximation which is frequently made in the literature [Lau, Jai]. This leads us to the study of a very simple energy functional

$$\mathcal{E}[\Psi] = N \int_{\mathbb{R}^2} V(z) \rho_\Psi(z) dz \tag{1.7}$$

depending only on the 1-particle probability density ρ_Ψ

$$\rho_\Psi(z) := \int_{\mathbb{R}^{2(N-1)}} |\Psi(z, z_2, \ldots, z_N)|^2 dz_2 \ldots dz_N \tag{1.8}$$

of the wave-function Ψ. Indeed, once the magnetic kinetic energy and the interaction energy are assumed to be fixed by the form (1.4) of the wave function, only the potential energy can vary non-trivially. In this paper we will be interested in studying the ground state energy

$$E(N) := \inf \left\{ \mathcal{E}[\Psi],\ \Psi \in \mathrm{Ker}(\mathcal{I}_N), \|\Psi\|_{L^2(\mathbb{R}^2)} = 1 \right\}. \tag{1.9}$$

In introducing this variational problem, which seems to be of a novel type, we have two main physical motivations in mind:

- In usual quantum Hall bars, electrons are confined and the disorder potential is crucial for the understanding of the physics of the quantum Hall effect. In Laughlin's original picture these aspects are neglected and only the translation-invariant case $V \equiv 0$ is considered. The finite size of the system is taken into account by fixing the filling factor, which amounts to imposing a maximum degree to the polynomial (holomorphic) part of the wave-function. The disorder is neglected altogether, and it is argued that the proposed wave-function is robust enough that these simplifications do not harm the conclusions in real samples, a fact amply confirmed by experiments (see [STG] and references therein). It is this fact that we wish to study by considering the variational problem described above. In this context, the pair-interaction is given by the 3D Coulomb kernel $w(\mathbf{x}-\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^{-1}$, so the Laughlin state is not an exact ground state. We shall, however, assume that the interaction is negligible when we work with functions of $\mathrm{Ker}(\mathcal{I}_N)$.

- It has been recognized for some time now ([Vie] and references therein) that the bosonic Laughlin state can in principle be created by fast rotation. Exploiting the analogy between the Coriolis and the Lorentz force, one may in this case identify the rotation frequency Ω around the axis perpendicular to the plane with an artificial magnetic field B. The potential V is then
$$V(\mathbf{x}) = V_{\mathrm{trap}}(\mathbf{x}) - \Omega^2 |\mathbf{x}|^2$$
where V_{trap} is the trapping potential confining the atoms, which is corrected to take the centrifugal force into account. For cold atomic gases, the interactions are short range and can be effectively modeled by contact potentials, for which the bosonic Laughlin state is an exact, zero-energy eigenstate [PB, LS, LSY, RSY2]. Experiments in these highly versatile systems (still elusive to this date) would allow an unprecedented direct probe of the properties of Laughlin's wave function. Recent experimental proposals [MF, RRD, Vie] involve engineering the trapping potential, which can lead to new physics [RSY1, RSY2]. The variational problem (1.9) is then intended as a way to study the shape imposed on the quantum Hall droplet by the trapping potential.

Motivated in particular by the two situations described above, our aim is to investigate the *incompressibility* properties of the Laughlin phase, in the form of a strong rigidity of its response to external potentials. Taking for granted the reduction to the LLL and cancellation of the interactions by the vanishing of the wave function along the diagonals of configuration space, we wish to see whether the Laughlin state, or a close variant, emerges as the natural ground state in a given potential. In particular, it is of importance to investigate the robustness of the correlations of Laughlin's wave function when the trapping potential is varied.

In view of (1.4) and (1.9), we are looking for a wave-function of the form
$$\Psi_F(z_1,\ldots,z_N) = c_F \Psi_{\mathrm{Lau}}(z_1,\ldots,z_N) F(z_1,\ldots,z_N)$$
where $F \in \mathcal{B}^N$ and c_F is a normalization factor. A natural *guess* is that, whatever the one-body potential, the correlations stay in the same form and the ground state is well-approximated by a wave function
$$\Psi(z_1,\ldots,z_N) = c_{f_1} \Psi_{\mathrm{Lau}}(z_1,\ldots,z_N) \prod_{j=1}^{N} f_1(z_j) \tag{1.10}$$
where the additional holomorphic factor F that characterizes functions of $\mathrm{Ker}(\mathcal{I}_N)$ is uncorrelated, which is the meaning of the ansatz $F = f_1^{\otimes N}$. Note that, although this is a natural guess (in the absence of interactions it does not seem favorable to correlate the state more than necessary), it is *far from being trivial*. Indeed, although (1.7) is a one-body functional in terms of Ψ, the correlation factor F really sees an effective, complicated many-body Hamiltonian because of the factor Ψ_{Lau} in Ψ_F.

The energy functional (1.7) is of course very simple and all the difficulty of the problem lies in the intricate structure of the variational set (1.4) of fully-correlated wave-functions. The expected rigidity of the strongly correlated states of (1.4) should manifest itself through the property that their densities are essentially bounded above by a universal constant

$$\rho_\Psi \lessapprox \frac{1}{\pi \ell N} \text{ for any } \Psi \in \mathrm{Ker}(\mathcal{I}_N). \tag{1.11}$$

This is the incompressibility notion we will investigate, in the limit $N \to \infty$. In view of existing numerical computations of the Laughlin state (e.g. [Cif]), (1.11) can hold only in some appropriate weak sense, see below. We are not able at present to study the full variational set (1.4) and we will make assumptions on the possible form of the additional correlation factor F in order to obtain a tractable model.

In fact we shall pursue along Laughlin's original intuition [Lau] and assume that particles are correlated only pairwise. This means that F contains only two-body correlation factors, i.e. that it can be written in the form

$$F(z_1, \dots, z_N) = \prod_{j=1}^{N} f_1(z_j) \prod_{(i,j) \in \{1,\dots,N\}} f_2(z_i, z_j) \tag{1.12}$$

with f_1 and f_2 two polynomials (f_2 being in addition a symmetric function of z, z') satisfying

$$\deg(f_1) \leq DN, \quad \deg(f_2) \leq D \tag{1.13}$$

for some constant D independent of N. The degree of $f_2(z, z')$ is here understood as the degree of the polynomial in z with z' fixed (and vice versa). Assumptions (1.12) and (1.13) are, of course, restrictive, but still cover a huge class of functions with possibly very intricate correlations and leads to a problem that is not at all trivial. They can also be relaxed to some extent, see Section 5.

Our *main result* is that any normalized state of $\mathrm{Ker}(\mathcal{I}_N)$

$$\Psi_F = c_F \, F \, \Psi_{\mathrm{Lau}}, \quad \|\Psi\|_{L^2(\mathbb{R}^2)} = 1 \tag{1.14}$$

corresponding to such correlation factors F satisfies (1.11) in the limit $N \to \infty$ in a suitable weak sense made clear below[3].

We then present some applications of theses estimates for the study of the variational problem (1.9), restricted to states of the above form. For a large class of radial potentials $V(|z|)$ we confirm the optimality of the ansatz (1.10): adding a correlations factor f_2 in (1.12) cannot reduce the energy. If the potential is increasing, the Laughlin state is always preferred, i.e. $f_1 \equiv 1$ gives the optimal energy. If the potential has a maximum at the origin, a situation we considered in [RSY1, RSY2], it is favorable to choose $f_1(z) = z^m$ and optimize over m. That one needs only use this form for a large class of radial potentials is an illustration of the expected rigidity of the Laughlin phase.

Remarks on terminology. We note that the notion of incompressibility we investigate is related to, but different from, the notion that there should be a gap in the energy spectrum above the energy of the Laughlin state. The latter notion depends on the Hamiltonian under consideration and it is not clear what it means when the Laughlin state is not an

[3] In particular we rescale the functions to take into account the fact that the right-hand side of (1.11) vanishes in the large N limit.

exact eigenstate, as e.g. in the case of Coulomb interaction. The two notions are in turn related to, but different from, the existence of a mobility gap.

In the sequel we will refer to states of $\mathrm{Ker}(\mathcal{I}_N)$ as "fully-correlated" since they include (at least) the strongly correlated factor $\Psi_{\rm Lau}$, and that adding more correlations does not decrease the interaction energy in contact potentials. States of the form (1.10) constitute the "Laughlin phase" in our language since they include exactly the correlations of the Laughlin state (1.3), the remaining factor describing i.i.d particles. Our goal is thus, assuming that states are fully correlated (which can be justified rigorously in some asymptotic regimes [LS, RSY1, RSY2]), to show that one can reduce to the Laughlin phase. We will call F a "correlation factor" since it encodes eventual correlations added to those of the Laughlin state, and the associated fully-correlated state $\Psi = c_F F \Psi_{\rm Lau}$ will always be understood to be normalized in L^2 thanks to the constant c_F.

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2. Main results

From now on we consider the Laughlin state (1.3) for a given, fixed integer ℓ, along with the associated set (1.4). As announced we focus on proving a particular case of incompressibility by considering special trial states. Recall the definition (1.5) of the N-body bosonic Bargmann space and define the set

$$\mathcal{V}_2^D = \Big\{ F \in \mathcal{B}^N \ : \ \text{there exist } (f_1, f_2) \in \mathcal{B} \times \mathcal{B}^2, \deg(f_1) \le DN, \deg(f_2) \le D, \\ F(z_1,\ldots,z_N) = \prod_{j=1}^N f_1(z_j) \prod_{1 \le i < j \le N} f_2(z_i,z_j) \Big\} \tag{2.1}$$

where we understand that f_1 and f_2 are polynomials, and we assume bounds on their degrees. Note that it is very natural to be less restrictive in the bound on the degree of f_1 than on the degree of f_2: taking some $f_2(z,z') = f_1(z) f_1(z')$ with $\deg(f_1) \le D/2$ we could construct a f_1 factor of degree DN anyway.

Given $F \in \mathcal{V}_2^D$ we define the corresponding fully-correlated state

$$\Psi_F(z_1,\ldots,z_N) = c_F \Psi_{\rm Lau}(z_1,\ldots,z_N) F(z_1,\ldots,z_N) \tag{2.2}$$

where c_F is a normalization factor ensuring $\|\Psi_F\|_{L^2(\mathbb{R}^2)} = 1$. It is a well-known fact [Lau, RSY2] that the one-particle density of the Laughlin state is approximately constant over a disc of radius $\sim \sqrt{N}$ and then quickly drops to 0. It is thus natural to also consider external potentials that live on this scale, which amounts to scale space variables and consider the energy functional

$$\mathcal{E}_N[\Psi] = (N-1) \int_{\mathbb{R}^2} V(\mathbf{x}) \, \rho_\Psi \left(\sqrt{N-1}\, \mathbf{x} \right) \tag{2.3}$$

where Ψ is of the form (2.2) and ρ_Ψ is the corresponding matter density. Note the choice of normalization: in view of (1.8),

$$\int_{\mathbb{R}^2} (N-1)\rho_\Psi \left(\sqrt{N-1}\,\mathbf{x}\right) d\mathbf{x} = 1.$$

That we scale lengths by a factor $\sqrt{N-1}$ instead of $\sqrt{N}$ is of course irrelevant for large N. It only serves to simplify some expressions in Section 3.1 below.

We define the ground state energy in the set $\mathcal{V}_2^D$ as

$$E_2^D(N) := \inf \left\{ \mathcal{E}_N[\Psi_F],\ \Psi_F \text{ of the form (2.2) with } F \in \mathcal{V}_2^D \right\}. \tag{2.4}$$

We can now state our incompressibility result, in the form of a universal lower bound on $E_2^D(N)$:

Theorem 2.1 (Weak incompressibility estimate for the Laughlin phase). *Let $V \in C^2(\mathbb{R}^2)$ be increasing at infinity in the sense that*

$$\min_{|x|\geq R} V(\mathbf{x}) \to \infty \quad \textit{for} \quad R \to \infty. \tag{2.5}$$

Define the corresponding "bath-tub energy"

$$E_V(\ell) := \inf \left\{ \int_{\mathbb{R}^2} V\rho \mid \rho \in L^1(\mathbb{R}^2),\ 0 \leq \rho \leq \frac{1}{\pi\ell},\ \int_{\mathbb{R}^2} \rho = 1 \right\}. \tag{2.6}$$

Then

$$\liminf_{N\to\infty} E_2^D(N) \geq E_V(\ell). \tag{2.7}$$

That the estimate (2.7) holds for any regular potential[4] increasing at infinity is a weak formulation of the bound (1.11). Indeed, observe that the energy in the potential V may not be lower than the minimal value (2.6) in the set of densities that satisfy (1.11) (note the rescaling of length and density units). It is well-known that the latter minimum energy is attained for a ρ saturating the imposed L^∞ bound, a fact that is usually refered to as the "bath-tub principle", see [LL, Theorem 1.14]. This weak version of a maximal density bound provided by Theorem 2.1 is a consequence of the two crucial characteristics of the FQHE: the restriction to the lowest Landau level and the strong, Laughlin-like, correlations. To clarify this, we make the following remarks.

Remark 2.2 (Illustrative comparisons).
How would the energy (2.3) behave in less constrained variational sets ? Three interesting cases are worth considering:

- *Particles outside the* LLL. Suppose the single-particle Hilbert space was the full space $L^2(\mathbb{R}^2)$ instead of the constrained space $\mathfrak{H}$. The minimization is then of course very simple and we would obtain $E_N = \min V$ by taking a minimizing sequence concentrating around a minimum point of V. Another way of saying this is that there is no upper bound whatsoever on the density of generic L^2 wave-functions, which is of course a trivial fact.

[4]The regularity assumption can be relaxed a bit.

- *Uncorrelated bosons in the* LLL. For non interacting bosons in a strong magnetic field, one should consider the space $\bigotimes_s^N \mathfrak{H}$, the symmetric tensor product of N copies of the LLL. The infimum in (2.3) can then be computed considering uncorrelated trial states of the form $f^{\otimes N}$, $f \in \mathfrak{H}$. LLL functions do satisfy a kind of incompressibility property because they are of the form holomorphic $\times$ gaussian. This can be made precise by the inequality [ABN2, Car, LS]
$$\sup_{z\in\mathbb{C}} \left| f(z) e^{-|z|^2/2} \right|^2 \leq \left\| f(.) e^{-|\,\cdot\,|/2} \right\|^2_{L^2(\mathbb{R}^2)}. \tag{2.8}$$
This is a much weaker notion however. (Compare (1.11) and (2.8), which leads to $\rho_\Psi \leq 1$.) In our scaled variables, one may still construct a sequence of the form $f^{\otimes N}$ concentrating around a minimum point of V without violating (2.8). The liminf in (2.7) is thus also equal to $\min V$ in this case.
- *Minimally correlated fermions in the* LLL. Due to the Pauli exclusion principle, fermions can never be uncorrelated: the corresponding wave functions have to be antisymmetric w.r.t. exchange of particles:
$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N) = -\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_N). \tag{2.9}$$
For LLL wave functions (which are continuous) this implies
$$\Psi(\mathbf{x}_i = \mathbf{x}_j) = 0 \text{ for any } i \neq j,$$
i.e. the wave function vanishes on the diagonals of the configuration space. Due to the holomorphy constraint, any N-body LLL fermionic wave-function is then of the form (2.2), where the Laughlin state is (1.3) with $\ell = 1$ and F has bosonic symmetry. If $F = f^{\otimes N}$, one could then talk of "minimally correlated" fermions[5].

  This case is covered by our theorem, and one obtains $E_V(1)$ as a lower bound to the energy. Observe that adding stronger correlations than those imposed by the Pauli principle, i.e. choosing an odd $\ell \geq 3$ increases the energy to the value $E_V(\ell)$ which is in general strictly larger than $E_V(1)$.

□

A very natural question is that of the optimality of Theorem 2.1. We can show that the lower bound (2.7) is in fact optimal when the potential is radial, increasing or mexican-hat-like, the infimum being in fact asymptotically reached in the Laughlin phase (1.10).

Corollary 2.3 (**Optimization of the energy in radial potentials**).
Let $V : \mathbb{R}^2 \mapsto \mathbb{R}$ be as in Theorem 2.1. Assume further that V is radial, has at most polynomial growth at infinity, and satisfies one of the two following assumptions

(1) *V is radially increasing,*
(2) *V has a "mexican-hat" shape: $V(|\mathbf{x}|)$ has a single local maximum at the origin and a single global minimum at some radius R.*

Then for D large enough it holds
$$\lim_{N\to\infty} E_2^D(N) = E_V(\ell). \tag{2.10}$$

[5]The Slater determinants for the Fermi sea at fixed total angular momentum can be obtained this way by taking $f(z) = z^m, m \in \mathbb{N}$. Note that for $\ell = 1$ the Laughlin state is a Vandermonde determinant and hence describes free fermions.

More precisely, in case (1)

$$\mathcal{E}_N[\Psi_{\mathrm{Lau}}] \to E_V(\ell) \textit{ when } N \to \infty \tag{2.11}$$

and in case (2) *one can find a fixed number* $\underline{m} \in \mathbb{R}$ *and a sequence* $m(N) \in \mathbb{N}$ *with* $m \sim \underline{m}N$ *as* $N \to \infty$ *such that, defining*

$$\Psi_m(z_1, \ldots, z_N) := c_m \Psi_{\mathrm{Lau}}(z_1, \ldots, z_N) \prod_{j=1}^{N} z_j^m \tag{2.12}$$

we have

$$\mathcal{E}_N[\Psi_m] \to E_V(\ell) \textit{ when } N \to \infty. \tag{2.13}$$

What this corollary says is that, for the particular potentials under consideration, the simplified ansatz (1.10) is optimal, at least amongst states built on $\mathcal{V}_2^D$. We *conjecture* that the latter restriction is unnecessary and that the result still holds when one considers the full variational set $\mathrm{Ker}(\mathcal{I}_N)$. Equations (2.11) and (2.13) are consequences of results obtained in [RSY2] on the densities of the Laughlin state and the Laughlin $\times$ giant vortex states (2.13): their densities in fact converge to the optimizer of the "bath-tub" energy (2.6). The remarkable fact about (2.11) is that the Laughlin state stays an approximate minimizer in any radially increasing potential. No matter how steep and narrow the imposed trapping potential, it is impossible to compress the Laughlin state while staying within the space $\mathrm{Ker}(\mathcal{I}_N)$. Also, (2.13) shows that if the potential has a radial well along some circle, the optimal density can be obtained by simply adding a vortex localized at the origin to the Laughlin state, and optimize over the added phase circulation.

Let us stress that, if Corollary 2.3 shows that the incompressibility upper bound can be saturated by some special states, other states of $\mathrm{Ker}(\mathcal{I}_N)$ can in fact have a significantly lower maximal density. This is for example the case of the states (2.12), when $m \gg N^2$, see again [RSY2] for details on this claim.

An interesting question concerns the generalization of the above result to more general potentials. For the simple form of the potentials we chose, it is not too difficult to construct a trial state whose density converges to the minimizer of the bath-tub energy. Constructing a trial state that does the same for a more general potential remains an *open problem*. We *conjecture* that a suitable trial state can always be constructed in the form (1.10), which would be a step towards confirming the robustness of Laughlin-like correlations in general potentials.

Before turning to the proofs of our results, let us comment on the physical relevance of the potentials considered in Corollary 2.3. See also the analysis of FQH interferometers in [LFS].

Remark 2.4 (Rotating trapped Bose gases).
The shape of the potentials considered in Corollary 2.3 is inspired from the experimental situation in rotating Bose gases, for which the relevant choice of Laughlin state is (1.3) with $\ell = 2$. Here the potential is usually radial and as we explained before, of the form

$$V(\mathbf{x}) = V_{\mathrm{trap}}(|\mathbf{x}|) - \Omega^2 |\mathbf{x}|^2.$$

In usual experiments the potential is to a good approximation quadratic, $V_{\mathrm{trap}}(r) = \Omega_{\mathrm{trap}}^2 r^2$, and it can be shown[6] that the Laughlin state is an exact ground state (in the

[6]The stability of the system requires of course V to be bounded below, and thus $\Omega_{\mathrm{trap}} \geq \Omega$.

case of a pure contact interaction). It seems, however, unlikely that the Laughlin state could be stabilized in such a potential, because the validity of the LLL approximation requires that $\Omega_{\rm trap}^2 - \Omega^2$ be extremely small, and thus the residual trapping, modified by the centrifugal force, to be extremely weak (see in particular [RRD] for a thorough discussion of experimental issues). To provide a better confinement against centrifugal forces, it has been proposed to use a steeper potential, a popular proposal being

$$V_{\rm trap}(r) = \Omega_{\rm trap}^2 r^2 + kr^4.$$

The effective potential taking the centrifugal force into account is then of the form

$$V(|\mathbf{x}|) = \omega N|\mathbf{x}|^2 + kN^2|\mathbf{x}|^4 \tag{2.14}$$

with space variables scaled as in (2.3) and $\omega = \Omega_{\rm trap}^2 - \Omega^2$. This potential is radial, increasing if $\omega > 0$ and mexican-hat like when $\omega < 0$. Corollary 2.3 applies directly in the case where $\omega = \omega_0 N^{-1}$ and $k = k_0 N^{-2}$ for fixed constants ω_0, k_0, and our methods can also accommodate more general choices. The previous results are thus a step towards the confirmation of a conjecture made in [RSY2], that states of the form (2.12) asymptotically optimize the potential energy in potentials of the form (2.14). This can, in principle, be checked experimentally (see again [RRD]): the states (2.12) all have a very rigid density profile, almost constant in a region and falling rapidly to 0 elsewhere [RSY2]. This differs markedly from the behavior of the Bose condensed phase, whose density follows closely the shape of the trap (see e.g. [ABD, ABN1, ABN2]), and may thus serve as an experimental probe of the Laughlin phase. More details on these aspects can be found in [RSY1, RSY2] and references therein. □

The rest of the paper contains the proofs of our main results. The core of the argument is in Section 3 where we argue that wave-functions built on $\mathcal{V}_2^D$, in fact satisfy more precise incompressibility estimates than stated in Theorem 2.1, with controlled error terms, see Theorem 3.1. Our main tool is Laughlin's plasma analogy, recalled below, and a new approach to the mean-field limit of classical Gibbs states. The method is based on a theorem by Diaconis and Freedman recalled in Appendix A, and we believe that it is of independent interest. The proof of Theorem 2.1 is concluded in Section 4.1. The additional ingredients needed for the proof of Corollary 2.3, taken from [RSY2], are recalled in Subsection 4.2. A final Section 5 explains possible generalizations of our results for correlation factors more complicated than (1.12).

3. Incompressibility estimates and the plasma analogy

3.1. **Main estimate.** In this section we consider a correlation factor $F \in \mathcal{V}_2^D$, that is we pick two functions $f_1, f_2 \in \mathcal{B}^1, \mathcal{B}^2$ and take F to be of the form

$$F(z_1, \dots, z_N) := \prod_{j=1}^N f_1(z_j) \prod_{1 \le i < j \le N} f_2(z_i, z_j)(z_i - z_j)^m. \tag{3.1}$$

Without loss (changing the value of the parameter $m \in \mathbb{N}$ if necessary), we may assume that f_2 is not identically 0 on the diagonal $z_i = z_j$, in which case the set $\{f_2(z_i, z_j) = 0\} \cap \{z_i = z_j\}$ is of dimension 0 (consists only of isolated points). Since the normalized wave-function (1.14) does not change when the polynomials f_1, f_2 are multiplied by constants, we may without loss assume that

$$
\begin{aligned}
|f_1(z)| &\leq |z|^{DN} + 1 \\
|f_2(z, z')| &\leq |z|^D + |z'|^D + 1
\end{aligned}
\tag{3.2}
$$

in view of our assumptions on their degrees.

Fixing some $\ell \in \mathbb{N}$, we want to analyze the density of the corresponding fully-correlated state (2.2), appearing in the energy functional (1.7). To this end we use Laughlin's plasma analogy [BCR, Lau, Lau2, LFS], regarding $|\Psi_F|^2$ as the Gibbs measure of a system of classical charged particles. We first scale variables by defining

$$
\mu_F(Z) := (N-1)^N \left| \Psi_F \left(\sqrt{N-1}\, Z \right) \right|^2 \tag{3.3}
$$

and we can then write μ_F as ($\mathcal{Z}_F \in \mathbb{R}$ normalizes the function in L^1)

$$
\mu_F(Z) = \frac{1}{\mathcal{Z}_F} \exp\left(-\frac{1}{T} H_F(Z) \right) \tag{3.4}
$$

where the temperature is

$$
T = (N-1)^{-1}
$$

and the Hamiltonian function H_F is

$$
\begin{aligned}
H_F(Z) = \sum_{j=1}^N \left(|z_j|^2 - \frac{2}{N-1} \log |g_1(z_j)| \right) \\
+ \frac{2}{N-1} \sum_{1 \leq i < j \leq N} \left(-(\ell + m) \log |z_i - z_j| - \log |g_2(z_i, z_j)| \right)
\end{aligned}
\tag{3.5}
$$

with

$$
g_1(z) = f_1\left(\sqrt{N-1} z \right), \quad g_2(z, z') = f_2\left(\sqrt{N-1} z, \sqrt{N-1} z' \right). \tag{3.6}
$$

This is a classical Hamiltonian with mean-field pair interactions: the factor $(N-1)^{-1}$ multiplying the 2-body part, due to the scaling in (3.3), makes the interaction energy of the same order of magnitude as the one-body energy. This is crucial for extracting information from the plasma analogy in the limit $N \to \infty$. Here we use in an essential manner the structure "holomorphic $\times$ gaussian" of LLL functions.

One may interpret (3.5) as the energy of N charged point particles located at $z_1, \ldots, z_N \in \mathbb{R}^2$ under the influence of the following potentials:

- A confining harmonic electrostatic potential (the $|z_j|^2$ term in the one-body part).
- The potential generated by fixed charges located at the zeros of the function g_1 (the $-\log |g_1(z_j)|$ term in the one-body part).
- The usual 2D Coulomb repulsion between particles of same charge (the $-(\ell + m) \log |z_i - z_j|$ terms in the two-body part).
- The Coulomb repulsion due to phantom charges located at the zeros of $g_2(z_i, z_j)$ (the $-\log |g_2(z_i, z_j)|$ terms in the two-body part). This term describes an intricate two-body interaction and it is responsible for most of the difficulties we encounter below. Particle i feels an additional charge attached to each other particle j in a complicated manner encoded in the zeros of $g_2(z_i, z_j)$. When particle j moves in

a straight line, the phantom charges attached to it may follow any algebraic curve in the plane.

By definition, μ_F minimizes the free energy functional

$$\mathcal{F}[\mu] := \mathcal{E}[\mu] + T \int_{\mathbb{R}^{2N}} \mu(Z) \log(\mu(Z)) dZ \tag{3.7}$$

with the energy term

$$\mathcal{E}[\mu] := \int_{\mathbb{R}^{2N}} H_F(Z)\mu(Z) dZ \tag{3.8}$$

on the set $\mathcal{P}(\mathbb{R}^{2N})$ of probability measures on $\mathbb{R}^{2N}$, and we have the relation

$$\mathcal{F}[\mu_F] = \inf_{\mu \in \mathcal{P}(\mathbb{R}^{2N})} \mathcal{F}[\mu] = -T \log \mathcal{Z}_F = -\frac{\log \mathcal{Z}_F}{N-1}. \tag{3.9}$$

Our goal here is to relate, in the large N limit, the N-body minimization problem (3.9) to the analogous problem for the mean-field energy functional defined as

$$\begin{aligned}\mathcal{E}^{\rm MF}[\rho] := \int_{\mathbb{R}^2} \left(|z|^2 - \frac{2}{N-1} \log |g_1(z)| \right) \rho(z) dz \\ + \int_{\mathbb{R}^4} \rho(z) \big(-(\ell+m) \log |z-z'| - \log |g_2(z,z')| \big) \rho(z') dz dz'\end{aligned} \tag{3.10}$$

for a probability measure $\rho \in \mathcal{P}(\mathbb{R}^2)$. We shall denote by $E^{\rm MF}, \varrho^{\rm MF}$ the minimum of the functional (3.10) and a minimizer respectively. Both may be proved to exist by standard arguments. More precisely, introducing for any symmetric $\mu \in \mathcal{P}_{\rm sym}(\mathbb{R}^{2N})$ the k-marginals[7]

$$\mu^{(k)}(z_1, \ldots, z_k) := \int_{\mathbb{R}^{2(N-k)}} \mu(z_1, \ldots, z_k, z'_{k+1}, \ldots, z'_N) dz'_{k+1} dz'_N \tag{3.11}$$

we should expect that, when N is large,

$$\mu_F^{(1)} \approx \varrho^{\rm MF} \tag{3.12}$$

and more generally

$$\mu_F^{(k)} \approx \left(\varrho^{\rm MF}\right)^{\otimes k} \tag{3.13}$$

in some appropriate sense, which is essentially a consequence of an energy estimate of the form

$$-T \log \mathcal{Z}_F \approx N E^{\rm MF}. \tag{3.14}$$

The rationale behind these expectations is that, N being large and the pair-interactions in (3.5) scaled to contribute at the same level as the one-body part, one should expect an uncorrelated ansatz $\mu_F \approx \rho^{\otimes N}$ to be a reasonable approximation for the minimization problem (3.9). Then, since the temperature $T \to 0$ in this limit, the entropy term can be safely dropped as far as leading order considerations are concerned. With this ansatz and simplification, the minimization of $\mathcal{F}[\rho^{\otimes N}]$ reduces to that of $\mathcal{E}^{\rm MF}[\rho]$, so that one may expect (3.13) and (3.14) to be reasonable guesses.

[7]Or reduced k-body densities, or k-correlation functions. Note that by symmetry, the choice of the variables over which to integrate is not important.

The $k = 1$ version of (3.13) provides the incompressibility property we are after: As we will show below

$$\varrho^{\mathrm{MF}} \leq \frac{1}{\ell \pi} \tag{3.15}$$

independently of F. In fact, as we have shown in [RSY1, RSY2], the density of the Laughlin state saturates this bound in the sense that when $F \equiv 1$

$$\mu_F^{(1)} \approx \varrho^{\mathrm{MF}} = \begin{cases} \dfrac{1}{\ell\pi} \text{ in } \mathrm{supp}(\varrho^{\mathrm{MF}}) \\ 0 \text{ otherwise.} \end{cases}$$

We can thus be a little bit more precise by interpreting our results as saying that no additional correlation factor of the type considered can compress the Laughlin state.

The result we aim at in this section is the following version of incompressibility.

Theorem 3.1 (Incompressibility for states with two-body correlations).
Let μ_F be defined as above. Pick some test one-body potential U such that $U, \Delta U \in L^\infty(\mathbb{R}^2)$. For any N large enough and $\varepsilon > 0$ small enough there exists an absolutely continuous probability measure $\rho_F \in L^1(\mathbb{R}^2)$ satisfying

$$\rho_F \leq \frac{1}{\pi(\ell + m)} + \varepsilon \frac{\sup |\Delta U|}{4\pi(\ell + m)} \tag{3.16}$$

such that

$$\int_{\mathbb{R}^2} U \mu_F^{(1)} \geq \int_{\mathbb{R}^2} U \rho_F - C(N\varepsilon)^{-1} \mathrm{Err}(m, D, \varepsilon U) \tag{3.17}$$

where $\mathrm{Err}(m, D, \varepsilon U)$ *is a quantity satisfying*

$$|\mathrm{Err}(m, D, \varepsilon U)| \leq 1 + (\ell + m + D) \log N + D \log D + (1 + \varepsilon \sup |\Delta U|). \tag{3.18}$$

Before we present the proof of this result, several remarks are in order:

- The L^1 function ρ_F appearing in the theorem is in fact the solution to a variational problem for a perturbed functional related to (3.10), that we introduce below. The perturbation is responsible for the ε-dependent term in the right-hand side of (3.16).
- The upper bound (3.16) is a slightly weakened version of (3.15). In applications there is a trade-off between choosing ε very small so that the upper bound (3.16) is as close as possible to (3.15), and ε very large so that the error term in the lower bound (3.17) is as small as possible.
- One may wonder whether the lower bound (3.17) is optimal. Our method yields an upper bound in terms of a slightly different reference density, indicating that (3.17) is in fact optimal up to remainder terms, see Remark 3.6 below.
- We have made error terms explicit but what matters for the sequel is that
$$(N\varepsilon)^{-1} \mathrm{Err}(m, D, \varepsilon U) \to 0$$
when $N \to \infty$ and $1 \gg \varepsilon \gg N^{-1}$ with m, D and U being kept fixed.

We note also the following additions to Remark 2.2.

Remark 3.2 (Effect of correlations).
One may be surprised that the marginal densities of highly correlated trial states are efficiently approximated in the form $\rho^{\otimes k}$, which corresponds to independent and identically

distributed classical particles. There are several answers to this. First, as noted in Remark 2.2, the correlations are responsible for reducing substantially the maximum density. Then, one should note that although the marginal densities of $|\Psi_F|^2$ are safely approximated by $\rho^{\otimes k}$, the appropriate ρ is in general not the square of the modulus of a LLL function. The emergence of the appropriate density profile is thus also a consequence of the correlations. Finally, the fact that the density $|\Psi_F|^2$ factorizes is by no means an indication that the state itself is effectively uncorrelated. Indeed, much of FQHE physics is based on the correlations in the phase of the wave-function. It is of crucial importance [STG, Jai] that the correlations may be seen as due to each particle carrying quantized vortices, and thus phase singularities. □

The rest of this section contains the proof of Theorem 3.1 and related comments. The main idea is as follows : We want to investigate the large N-limit of the minimization problem (3.9) with the goal of justifying (3.13) and (3.14). This may look like a classical problem at first, but a little further thought reveals that existing methods are not enough to deal with it at the level of precision we require. First, note that the data of the problem, namely g_1, g_2 and m may depend on N, which rules out methods based on compactness arguments [MS, Kie, CLMP, KS]. Secondly, the two-body potential is not of the form $w(z - z')$ and is not of positive type[8], which is the crucial property on which known quantitative methods rely (see [RSY2, RS, SS] and references therein).

Our approach is based on a quantitative version of the Hewitt-Savage [HS] theorem. The theorem we use, due to Diaconis and Freedman [DF], is recalled in Appendix A for the convenience of the reader. It states that the k-marginals of any classical symmetric N-body state are close to convex combinations of uncorrelated states of the form $\rho^{\otimes k}$ when $k \ll \sqrt{N}$. In the regime we investigate, the temperature is so small that we may neglect the entropy term in (3.7). The free-energy functional (3.7) is thus essentially an affine functional of μ and the infimum is approximately reached on the extremal points of the convex set $\mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{2N})$. But the Diaconis-Freedman theorem essentially says that the marginals of the latter are all close to states of the form $\rho^{\otimes k}$, $\rho \in \mathcal{P}(\mathbb{R}^2)$, which explains why (3.14) should hold in reasonable generality.

3.2. **Preliminaries.** Our strategy requires two modifications of the Hamiltonian:

- First, the Diaconis-Freedman theorem can be put to good use only with two-body Hamiltonians that are not singular along the diagonals $z_i = z_j$. Our Hamiltonian (3.5) has Coulomb singularities, however, that we regularize by smearing out charges. The charge distribution of a unit charge smeared over a disc of radius α is
$$\delta_\alpha(z) := \begin{cases} \dfrac{1}{\pi\alpha^2} & \text{if } |z| \le \alpha \\ 0 & \text{if } |z| > \alpha. \end{cases} \tag{3.19}$$
It tends to the delta function as $\alpha \to 0$. The corresponding electrostatic potential is
$$-\log_\alpha |z| := -\int \log|z - w|\, \delta_\alpha(w) dw. \tag{3.20}$$

[8]i.e. of positive Fourier transform.

Since $\Delta \log|z| = 2\pi\delta(z)$ we have

$$\Delta \log_\alpha |z| = 2\pi\delta_\alpha(z). \tag{3.21}$$

Using Newton's theorem ([LL], Theorem 9.7 or [RSY2], Lemma 3.3) it is straightforward to compute

$$-\log_\alpha |z| = \begin{cases} -\log\alpha + \frac{1}{2}\left(1 - \frac{|z|^2}{\alpha^2}\right) & \text{if } |z| \le \alpha \\ -\log|z| & \text{if } |z| \ge \alpha \end{cases} \tag{3.22}$$

for any $z \in \mathbb{R}^2$. Clearly we have

$$-\log_\alpha |z| \le \max\left(-\log|z|, -\log\alpha + \frac{1}{2}\right) \tag{3.23}$$

and $-\log_\alpha |z|$ tends pointwise to $-\log|z|$ for $z \neq 0$ as $\alpha \to 0$. Moreover, (3.22) implies

$$-\frac{d}{d\alpha}\log_\alpha |z| = \begin{cases} -\frac{1}{\alpha} + \frac{|z|^2}{\alpha^3} & \text{if } |z| \le \alpha \\ 0 & \text{if } |z| \ge \alpha. \end{cases} \tag{3.24}$$

To obtain energy lower bounds we will replace the two-body potential

$$W(z,z') = -2(\ell+m)\log|z-z'| - 2\log|g_2(z,z')|$$

appearing in (3.5) by the regularized

$$W_\alpha(z,z') := -2(\ell+m)\log_\alpha|z-z'| - 2\log_\alpha\left|g_2(z,z')\right|, \tag{3.25}$$

recalling that, by (3.20),

$$\log_\alpha\left|g_2(z,z')\right| = \int \log\left|g_2(z,z') - w\right| \delta_\alpha(w)\,dw. \tag{3.26}$$

We shall also use that the log of the absolute value of a holomorphic function is subharmonic, so

$$\Delta \log_\alpha |g_1(z)| \ge 0 \quad \text{ and } \quad \Delta_z \log_\alpha |g_2(z,z')| \ge 0. \tag{3.27}$$

Finally, for $\alpha = 0$ we define

$$\log_0 = \log \quad \text{ and } \quad \delta_0(z) = \delta(z). \tag{3.28}$$

- The second modification of the Hamiltonian is due to the following. To show that (3.12) follows from (3.14), the usual way is to vary the one-body potential in the Hamiltonian and use the Feynman-Hellmann principle to obtain a weak-$*$ convergence in L^1. Here we are looking for quantitative estimates on objects that might depend on N even in the mean-field approximation, so weak convergence arguments are not available. We thus implement the variation explicitly, adding a weak one-body potential εU to the Hamiltonian, with ε a (small) number and $U \in L^\infty(\mathbb{R}^2)$ being the potential appearing in Theorem 3.1. Analyzing the mean-field limits at $\varepsilon = 0$ and $\varepsilon \neq 0$ small enough will give information on the one-body density at $\varepsilon = 0$, i.e. for the original problem.

These considerations lead us to the following modified Hamiltonian:

$$H_{\varepsilon,\alpha}(z_1,\dots,z_N) := \sum_{j=1}^N \left(|z_j|^2 - \frac{2}{N-1}\log|g_1(z_j)| + \varepsilon U(z_j) \right) + \frac{1}{N-1}\sum_{1\le i<j\le N} W_\alpha(z_i,z_j) \quad (3.29)$$

to which we associate the free-energy functional (still with $T=(N-1)^{-1}$ and $\mu \in \mathcal{P}(\mathbb{R}^{2N})$)

$$\mathcal{F}_{\varepsilon,\alpha}[\mu] := \int_{Z\in\mathbb{R}^{2N}} H_{\varepsilon,\alpha}(Z)\mu(Z)dZ + T\int_{Z\in\mathbb{R}^{2N}} \mu(Z)\log(\mu(Z))dZ. \quad (3.30)$$

We denote by

$$\mu_{\varepsilon,\alpha}(Z) := \frac{1}{\mathcal{Z}_{\varepsilon,\alpha}} \exp\left(-\frac{1}{T}H_{\varepsilon,\alpha}(Z)\right)$$

the associated Gibbs measure, and we have

$$\mathcal{F}_{\varepsilon,\alpha}[\mu_{\varepsilon,\alpha}] = -T\log\mathcal{Z}_{\varepsilon,\alpha} = \inf_{\mu\in\mathcal{P}(\mathbb{R}^{2N})} \mathcal{F}_{\varepsilon,\alpha}[\mu].$$

We will relate the minimization of (3.30) to that of the mean-field energy functional

$$\mathcal{E}^{\rm MF}_{\varepsilon,\alpha}[\rho] := \int_{\mathbb{R}^2} \left(|z|^2 - \frac{2}{N-1}\log|g_1(z)| + \varepsilon U(z) \right)\rho(z)dz + \frac{1}{2}\int_{\mathbb{R}^4} \rho(z)W_\alpha(z,z')\rho(z')dzdz' \quad (3.31)$$

with minimum $E^{\rm MF}_{\varepsilon,\alpha}$ and minimizer $\varrho^{\rm MF}_{\varepsilon,\alpha}$ (amongst probability measures). With this notation, the original problem we are ultimately interested in corresponds to the choice $\varepsilon=\alpha=0$.

As a preparation for the study of the MF limit below, we investigate the properties of this family of mean-field functionals. In particular we prove the upper bound (3.15), and show that some weak version of it survives the introduction of the small parameters α and ε.

Lemma 3.3 **(The mean-field densities).**
For any ε,α small enough there exists at least one $\varrho^{\rm MF}_{\varepsilon,\alpha}$ minimizing (3.31) *amongst probability measures. Any minimizer satisfies the variational equation*

$$\int_{\mathbb{R}^2} W_\alpha(z,z')\rho(z')dz' + \left(|z|^2 - \frac{2}{N-1}\log|g_1(z)| + \varepsilon U\right) \begin{cases} = E^{\rm MF}_{\varepsilon,\alpha} + \frac12\int_{\mathbb{R}^4}\varrho^{\rm MF}_{\varepsilon,\alpha}(z)W_\alpha(z,z')\varrho^{\rm MF}_{\varepsilon,\alpha}(z')dzdz' \text{ if } z\in\operatorname{supp}(\varrho^{\rm MF}_{\varepsilon,\alpha}) \\ \ge E^{\rm MF}_{\varepsilon,\alpha} + \frac12\int_{\mathbb{R}^4}\varrho^{\rm MF}_{\varepsilon,\alpha}(z)W_\alpha(z,z')\varrho^{\rm MF}_{\varepsilon,\alpha}(z')dzdz' \text{ otherwise.}\end{cases} \quad (3.32)$$

Moreover, we have the following bounds:

- *(Case $\alpha=0$). For any $z\in\mathbb{R}^2$*

$$\varrho^{\rm MF}_{\varepsilon,0}(z) \le \frac{1}{\pi(\ell+m)} + \varepsilon\frac{\sup|\Delta U|}{4\pi(\ell+m)} \quad (3.33)$$

- *(Case $\alpha > 0$). For any $z \in \mathbb{R}^2$*

$$\delta_\alpha * \varrho_{\varepsilon,\alpha}^{\mathrm{MF}}(z) \leq \frac{1}{\pi(\ell+m)} + \varepsilon \frac{\sup|\Delta U|}{4\pi(\ell+m)} \tag{3.34}$$

Inequality (3.15) in the case $F \equiv 1$ is simply (3.33) with $m = 0$ and $\varepsilon = 0$.

Proof. The existence part and the variational equation follow by standard methods, see [ST, Chapter 1]. In particular, note that the confining potential guarantees the tightness of minimizing sequences. The bounds in (3.33) and (3.34) are deduced by applying the Laplacian to the variational equation (3.32), using that

$$\Delta \log_\alpha |z - z'| = 2\pi\delta_\alpha(z - z'),\ \Delta|z|^2 = 4$$

and

$$\Delta \log_\alpha |g_1(z)| \geq 0,\ \Delta_z \log_\alpha |g_2(z, z')| \geq 0.$$

The latter inequalities follow from the fact that g_1 and g_2 are holomorphic functions. □

In view of the properties of our regularized interaction potentials, it is clear that $E_{\varepsilon,\alpha}^{\mathrm{MF}}$ approximates $E_{\varepsilon,0}^{\mathrm{MF}}$ in the limit $\alpha \to 0$, which is the content of the following:

Lemma 3.4 (**Small α limit of the mean-field energy**).
For any ε, α small enough the following bound holds:

$$E_{\varepsilon,\alpha}^{\mathrm{MF}} \leq E_{\varepsilon,0}^{\mathrm{MF}} \leq E_{\varepsilon,\alpha}^{\mathrm{MF}} + C\left(1 + \varepsilon \sup|\Delta U|\right) \alpha^{\min(2,2/D)} \tag{3.35}$$

where D is the degree of $g_2(z, z')$ as a function of either z or z'.

Proof. The lower bound $E_{\varepsilon,\alpha}^{\mathrm{MF}} \leq E_{\varepsilon,0}^{\mathrm{MF}}$ is an obvious consequence of $-\log_\alpha |z| \leq -\log|z|$. For the upper bound we use the formula

$$E_{\varepsilon,\alpha}^{\mathrm{MF}} - E_{\varepsilon,0}^{\mathrm{MF}} = \int_0^\alpha \frac{d}{d\alpha'} E_{\varepsilon,\alpha'}^{\mathrm{MF}} d\alpha'. \tag{3.36}$$

We evaluate $\frac{d}{d\alpha'} E_{\varepsilon,\alpha'}^{\mathrm{MF}}$ by means of the Feynman-Hellman theorem:

$$\frac{d}{d\alpha'} E_{\varepsilon,\alpha'}^{\mathrm{MF}} = \left[\frac{d}{d\alpha'} \mathcal{E}_{\varepsilon,\alpha'}^{\mathrm{MF}}\right] [\rho_{\varepsilon,\alpha'}^{\mathrm{MF}}], \tag{3.37}$$

and, for any ρ,

$$\left[\frac{d}{d\alpha'} \mathcal{E}_{\varepsilon,\alpha'}^{\mathrm{MF}}\right] [\rho] = \frac{1}{2} \iint \rho(z) \frac{d}{d\alpha'} W_{\alpha'}(z, z') \rho(z') dz dz'. \tag{3.38}$$

Moreover, by (3.24) we have

$$\left| \frac{d}{d\alpha'} W_{\alpha'}(z, z') \right| \leq \frac{1}{\alpha'}(\ell + m + 1) \text{ if } |z - z'| \leq \alpha' \text{ or } |g_2(z, z')| \leq \alpha' \tag{3.39}$$

and 0 otherwise.

Next we note that as a consequence of (3.34), we have for any set Λ of area $|\Lambda|$ that can be covered by $O(|\Lambda|/\alpha^2)$ discs of radius α

$$\int_\Lambda \rho_{\varepsilon,\alpha}^{\mathrm{MF}}(z) dz \leq C\, \frac{1 + \varepsilon \sup|\Delta U|}{\ell + m}\, |\Lambda|. \tag{3.40}$$

Since $g_2(z,z')$ has a factorization

$$g_2(z,z') = \prod_{i=1}^{D} (z - c_i(z')) \tag{3.41}$$

where the zeros $c_i(z')$ are algebraic functions of z', it is easy to see that for any z' the total area of the set of z's where $|z - z'| \leq \alpha'$ or $|g_2(z,z')| \leq \alpha'$ is bounded by

$$\begin{cases} C(\alpha')^{2/D} \text{ if } D \geq 1 \\ C(\alpha')^2 \text{ if } D = 0 \end{cases}$$

where C and D depend only on g_2 but are independent of z'. Moreover this set may be covered by $O(\alpha^{2/D-2})$ discs of radius α^2 in the first case and $O(1)$ discs in the second case. We may then employ (3.40) on it.

Combining (3.38), (3.39), (3.40), and the normalization $\int \rho^{\mathrm{MF}}_{\varepsilon,\alpha'} = 1$, we obtain

$$\left| \frac{d}{d\alpha'} E^{\mathrm{MF}}_{\varepsilon,\alpha'} \right| \leq C \left(1 + \varepsilon \sup |\Delta U|\right) (\alpha')^{\min(2,2/D)}. \tag{3.42}$$

Integrating from 0 to α then gives (3.35).

□

3.3. **Mean-field/small temperature limit.** The crux of our method is the proof of the following lower bounds for our family of many-body free energies. The corresponding upper bounds are easy to derive, using the usual $\rho^{\otimes N}$ ansatz, but they shall not concern us at this stage.

Proposition 3.5 (**Free-energy lower bounds**).
Under the previous assumptions, the following holds for any α, ε small enough and $\mu \in \mathcal{P}(\mathbb{R}^{2N})$:

$$\begin{aligned} \mathcal{F}_{\varepsilon,\alpha}[\mu] \geq N E^{\mathrm{MF}}_{\varepsilon,\alpha} &- C\left(\ell + m + D\right)\left(|\log \alpha| + 1\right) \\ &- C\left(D \log D + D \log N + (\ell + m)\log(\ell + m)\right). \end{aligned} \tag{3.43}$$

Proof. In view of the symmetry of the Hamiltonian (3.29), we may consider the minimization restricted to symmetric probabilities. For such a $\mu \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{2N})$ the free-energy (3.30) may be rewritten

$$\begin{aligned} \mathcal{F}_{\varepsilon,\alpha}[\mu] = N \int_{\mathbb{R}^2} \mu^{(1)}(z) \left(|z|^2 - \frac{2}{N-1} \log |g_1(z)| + \varepsilon U(z) \right) dz \\ + \frac{N}{2} \int_{\mathbb{R}^4} \mu^{(2)}(z,z') W_\alpha(z,z') dz dz' + \frac{1}{N-1} \int_{Z \in \mathbb{R}^{2N}} \mu(Z) \log(\mu(Z)) dZ \end{aligned}$$

with $\mu^{(1)}$ and $\mu^{(2)}$ the first and second marginals of μ, defined as in (3.11). We first deal with the entropy term: by positivity of relative entropies (see e.g. Lemma 3.1 in [RSY2])

$$\int_{Z \in \mathbb{R}^{2N}} \mu(Z) \log(\mu(Z)) dZ \geq \int_{Z \in \mathbb{R}^{2N}} \mu(Z) \log(\nu(Z)) dZ$$

for any probability measure ν. We set

$$\nu_0(z) = c_0 \exp(-|z|^2)$$

with c_0 a normalization constant and apply the above inequality with $\nu = \nu_0^{\otimes N}$. Integrating over $N-1$ variables we then obtain

$$\begin{aligned}\int_{Z\in\mathbb{R}^{2N}} \mu(Z)\log(\mu(Z))dZ &\geq N\int_{z\in\mathbb{R}^2} \mu^{(1)}(z)\log(\nu_0(z))dz\\ &= N\log(c_0) - N\int_{z\in\mathbb{R}^2} |z|^2\mu^{(1)}(z)dz\end{aligned}$$

which gives the lower bound

$$\begin{aligned}\mathcal{F}_{\varepsilon,\alpha}[\mu] \geq N\int_{\mathbb{R}^2} \mu^{(1)}(z)\left((1-N^{-1})|z|^2 - \frac{2}{N-1}\log|g_1(z)| + \varepsilon U(z)\right)dz\\ + \frac{N}{2}\int_{\mathbb{R}^4}\mu^{(2)}(z,z')W_\alpha(z,z')dzdz' - C \qquad (3.44)\end{aligned}$$

for any $\mu \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{2N})$. Now we apply to μ the construction of [DF] recalled in Appendix A. This gives a $P_\mu \in \mathcal{P}(\mathcal{P}(\mathbb{R}^2))$, a Borel probability measure over the probability measures of $\mathbb{R}^2$, and a

$$\tilde{\mu} := \int_{\rho\in\mathcal{P}(\mathbb{R}^2)} \rho^{\otimes N} P_\mu(d\rho) \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{2N}) \qquad (3.45)$$

such that, using (A.8),

$$\begin{aligned}\mu^{(1)}(z) &= \tilde{\mu}^{(1)}(z)\\ \mu^{(2)}(z,z') &= \frac{N}{N-1}\tilde{\mu}^{(2)}(z,z') - \frac{1}{N-1}\tilde{\mu}^{(1)}(z)\delta(z-z')\\ &= \tilde{\mu}^{(2)}(z,z') + \frac{1}{N-1}\left[\tilde{\mu}^{(2)}(z,z') - \tilde{\mu}^{(1)}(z)\delta(z-z')\right]. \qquad (3.46)\end{aligned}$$

It is at this point that it proves useful to have regularized the Coulomb part of the two-body interaction: since W_α is locally bounded from above we can insert (3.46) in (3.44). We then obtain

$$\begin{aligned}\mathcal{F}_{\varepsilon,\alpha}[\mu] \geq N\Bigg[\int_{\mathbb{R}^2} \tilde{\mu}^{(1)}(z)\left((1-2N^{-1})|z|^2 - \frac{2}{N-1}\log|g_1(z)| + \varepsilon U(z)\right)dz\\ +\frac{1}{2}\int_{\mathbb{R}^4}\tilde{\mu}^{(2)}(z,z')W_\alpha(z,z')dzdz'\Bigg]\\ + \int_{\mathbb{R}^4}\left[(|z|^2+|z'|^2)/2 + \frac{N}{2(N-1)}W_\alpha(z,z')\right]\tilde{\mu}^{(2)}(z,z')dzdz'\\ - \frac{N}{2(N-1}\int_{\mathbb{R}^2}\tilde{\mu}^{(1)}(z)W_\alpha(z,z) - C. \qquad (3.47)\end{aligned}$$

The last two lines are error terms, and in the second to last line we have borrowed

$$\int_{\mathbb{R}^2}|z|^2\tilde{\mu}^{(1)}(z)dz = \frac{1}{2}\int_{\mathbb{R}^4}(|z|^2+|z'|^2)\tilde{\mu}^{(2)}(z,z')dzdz' \qquad (3.48)$$

from the main term in the first two lines; this account for the $-2N^{-1}|z|^2$ rather than $-N^{-1}|z|^2$ in the main term.

The error term in the second to last line is easily estimated, using (3.2) and (3.22):

$$
\begin{aligned}
W_\alpha(z,z') &\geq -2(\ell+m)\log\left(|z|+|z'|\right) - 2\log\left(N^{D/2}\left(|z|^D+|z'|^D\right)+1\right)\\
&\geq -(\ell+m)\left(\log|z|+\log|z'|\right) - D\log N - 2\log\left(|z|^D+1\right) - 2\log\left(|z|^D+1\right)
\end{aligned}
\tag{3.49}
$$

leads to

$$
\begin{aligned}
\int_{\mathbb{R}^4}\left[\frac{1}{2}(|z|^2+|z'|^2)+\frac{N}{2(N-1)}W_\alpha(z,z')dzdz'\right]\tilde{\mu}^{(2)}(z,z')dzdz'\\
\geq -C(\ell+m)\log(\ell+m) - CD\log N - CD\log D.
\end{aligned}
\tag{3.50}
$$

The term in the last line of (3.47) is estimated as follows. Since $\tilde{\mu}$ is a probability, so is $\tilde{\mu}^{(1)}$ and we have

$$
\int_{\mathbb{R}^2}\tilde{\mu}^{(1)}(z)\left((\ell+m)\log_\alpha(0)+\log_\alpha|g_2(z,z)|\right) \geq C(\ell+m+D)\left(\log\alpha+\frac{1}{2}\right), \tag{3.51}
$$

where we used (3.22).

Inserting now the representation (3.45) into the main term proportional to N in (3.47) we see that this part can be written as

$$
N\int_{\rho\in\mathcal{P}(\mathbb{R}^2)}\tilde{\mathcal{E}}[\rho] \geq N\tilde{E} \tag{3.52}
$$

where $\tilde{\mathcal{E}}$ is the modified mean-field functional

$$
\begin{aligned}
\tilde{\mathcal{E}}[\rho] := \int_{\mathbb{R}^2}\left((1-2N^{-1})|z|^2 - \frac{2}{N-1}\log|g_1(z)| + \varepsilon U(z)\right)\rho(z)dz\\
+\frac{1}{2}\iint_{\mathbb{R}^4}\rho(z)W_\alpha(z,z')\rho(z')dz'
\end{aligned}
$$

and $\tilde{E}$ its infimum over $\mathcal{P}(\mathbb{R}^2)$. We can then use a minimizer $\tilde{\rho}$ for this functional and obtain

$$
\tilde{E} = \mathcal{E}^{\mathrm{MF}}_{\varepsilon,\alpha}[\tilde{\rho}] - 2N^{-1}\int_{\mathbb{R}^2}|z|^2\tilde{\rho} \geq E^{\mathrm{MF}}_{\varepsilon,\alpha} - 2N^{-1}\int_{\mathbb{R}^2}|z|^2\tilde{\rho}. \tag{3.53}
$$

The last term is again an error term, but up to this term and the error term give by (3.50) we have the desired lower bound. The final step is thus to estimate the term $2N^{-1}\int_{\mathbb{R}^2}|z|^2\tilde{\rho}$.

To this end we may on the one hand write

$$
\begin{aligned}
\tilde{\mathcal{E}}[\rho] = \int_{\mathbb{R}^4}dzdz'\rho(z)\rho(z')\Big[\frac{1}{2}W_\alpha(z,z')\\
+\frac{1}{2}\left((1-2N^{-1})(|z|^2+|z'|^2) - \frac{2}{N-1}\left(\log|g_1(z)|+\log|g_1(z')|\right)+\varepsilon U(z)+\varepsilon U(z')\right)\Big]
\end{aligned}
$$

to obtain

$$
\begin{aligned}
\tilde{\mathcal{E}}[\tilde{\rho}] &\geq \left(\frac{1}{2}-2N^{-1}\right)\int_{\mathbb{R}^2}|z|^2\tilde{\rho} + \inf_{\mathbb{R}^4}\tilde{W}\\
&\geq \frac{1}{2}\int_{\mathbb{R}^2}|z|^2\tilde{\rho} - C\left(1+D\log D+D\log N+(\ell+m)\log(\ell+m)\right)
\end{aligned}
$$

where $\tilde{W}$ is the two-body potential

$$\tilde{W}(z,z') = \frac{1}{4}\left(|z|^2+|z'|^2\right) - \frac{1}{2(N-1)}\left(\log|g_1(z)| + \log|g_1(z')|\right) + \frac{1}{2}W_\alpha(z,z')$$

and a lower bound to its infimum is derived by elementary considerations similar to (3.49). On the other hand, using as trial state the normalized characteristic function of a ball $B(0,1)$ centered at 0 of radius 1,

$$\rho^{\mathrm{trial}} = \frac{1}{|B(0,1)|}\mathbb{1}_{B(0,1)}$$

we easily have

$$\tilde{E} \le \tilde{\mathcal{E}}[\rho^{\mathrm{trial}}] \le C\left(1+D+\ell+m\right)$$

from which

$$\int_{\mathbb{R}^2} |z|^2 \tilde{\rho} \le C\left(1 + D\log(D) + (\ell+m)\log(\ell+m)\right) + D\log N \tag{3.54}$$

follows.

□

3.4. **Conclusion: proof of Theorem 3.1.** We recall that the μ_F we are interested in is equal to $\mu_{0,0}$ in the notation of Subsection 3.2. Let $\varepsilon > 0$ be chosen small enough that the results of Sections 3.2 and 3.3 may be applied. We first write

$$\begin{aligned} NE^{\mathrm{MF}}_{0,0} + N\varepsilon\int_{\mathbb{R}^2} U\mu^{(1)}_{0,0} + C &\ge \mathcal{F}_{0,0}[\mu_{0,0}] + N\varepsilon\int_{\mathbb{R}^2} U\mu^{(1)}_{0,0} \\ &\ge \mathcal{F}_{\varepsilon,0}[\mu_{\varepsilon,0}] \ge \mathcal{F}_{\varepsilon,\alpha}[\mu_{\varepsilon,0}] \ge \mathcal{F}_{\varepsilon,\alpha}[\mu_{\varepsilon,\alpha}]. \end{aligned} \tag{3.55}$$

The first inequality is proved using $(\varrho^{\mathrm{MF}}_{0,0})^{\otimes N}$ as a trial state for $\mathcal{F}_{0,0}$. The entropy term

$$T\int_{\mathbb{R}^{2N}} (\varrho^{\mathrm{MF}}_{0,0})^{\otimes N} \log\left((\varrho^{\mathrm{MF}}_{0,0})^{\otimes N}\right) = NT\int_{\mathbb{R}^2} \varrho^{\mathrm{MF}}_{0,0}\log\varrho^{\mathrm{MF}}_{0,0}$$

is bounded above using (3.33) and recalling that the temperature is of order N^{-1}. The other inequalities in (3.55) use either the variational principle or $-\log|z| \ge -\log_\alpha|z|$.

Next, we use first Proposition 3.5 and then Lemma 3.4 to obtain

$$\begin{aligned} \mathcal{F}_{\varepsilon,\alpha}[\mu_{\varepsilon,\alpha}] &\ge NE^{\mathrm{MF}}_{\varepsilon,\alpha} - C\left(\ell+m+D\right)\left(|\log\alpha|+1\right) \\ &\quad - C\left(D\log D + D\log N + (\ell+m)\log(\ell+m)\right) \\ &\ge NE^{\mathrm{MF}}_{\varepsilon,0} - C\left(\ell+m+D\right)\left(|\log\alpha|+1\right) - CN(1+\varepsilon\sup|\Delta U|)\,\alpha^{\min(2,2/D)} \\ &\quad - C\left(D\log D + D\log N + (\ell+m)\log(\ell+m)\right). \end{aligned}$$

Then

$$E^{\mathrm{MF}}_{\varepsilon,0} = \mathcal{E}^{\mathrm{MF}}_{0,0}[\varrho^{\mathrm{MF}}_{\varepsilon,0}] + \varepsilon\int_{\mathbb{R}^2} U\varrho^{\mathrm{MF}}_{\varepsilon,0} \ge E^{\mathrm{MF}}_{0,0} + \varepsilon\int_{\mathbb{R}^2} U\varrho^{\mathrm{MF}}_{\varepsilon,0}$$

by the variational principle applied to $\mathcal{E}^{\mathrm{MF}}_{0,0}$. Summing up we have (recall that $\varepsilon > 0$)

$$
\begin{aligned}
\int_{\mathbb{R}^2} U \mu^{(1)}_{0,0} \geq \int_{\mathbb{R}^2} U \varrho^{\mathrm{MF}}_{\varepsilon,0}
&- C(N\varepsilon)^{-1} \left(1 + (\ell + m + D)\left(|\log \alpha| + 1\right) + N(1 + \varepsilon \sup |\Delta U|)\, \alpha^{\min(2,2/D)}\right) \\
&- C(N\varepsilon)^{-1} \left(D \log D + D \log N + (\ell + m)\log(\ell + m)\right) \\
&\geq \int_{\mathbb{R}^2} U \varrho^{\mathrm{MF}}_{\varepsilon,0} - C(N\varepsilon)^{-1} \left(1 + (\ell + m + D)\log N\right) \qquad (3.56) \\
&- C(N\varepsilon)^{-1} \left(D \log D + D \log N + (1 + \varepsilon \sup |\Delta U|)\right) \qquad (3.57)
\end{aligned}
$$

where we have chosen $\alpha = N^{-D/2}$ if $D \geq 1$ and $\alpha = N^{-1/2}$ if $D = 0$ to obtain the last inequality. Recalling that $\mu_{0,0} = \mu_F$, the lower bound (3.17) is proved, with $\rho_F := \varrho^{\mathrm{MF}}_{\varepsilon,0}$. This density satisfies (3.16) by Lemma 3.3.

□

Remark 3.6 (**The opposite inequality**).
Following the very same steps as above but using a perturbation potential $-\varepsilon U$ instead of εU in the Hamiltonian one obtains

$$
\int_{\mathbb{R}^2} U \mu^{(1)}_F \leq \int_{\mathbb{R}^2} U \hat{\rho}_F + C\,(N\varepsilon)^{-1}\, \mathrm{Err}(m, D, \varepsilon U). \qquad (3.58)
$$

with $\hat{\rho}_F := \varrho^{\mathrm{MF}}_{-\varepsilon,0}$. This density also satisfies (3.16) and the above estimate is thus a kind of converse to (3.17). Note that the reference density is different however. We shall not use this remark anywhere in the paper. □

4. Conclusion of the proofs

Here we conclude the proofs of Theorem 2.1 and Corollary 2.3, in Subsections 4.1 and 4.2 respectively.

4.1. **Response to external potentials: proof of Theorem 2.1.** We now bound from below the energy $\mathcal{E}_N[\Psi_F]$ when $F \in \mathcal{V}^D_2$ as defined in (2.1). Without loss, we write F as in (3.1), with

$$
m + \deg(f_2) \leq D \text{ and } \deg(f_1) \leq DN. \qquad (4.1)
$$

and assume that (3.2) holds. We will apply the analysis of Section 3. Let us pick a large constant B (to be tuned later on) and define the truncated potential

$$
V_B(\mathbf{x}) := \min\{V(\mathbf{x}), B\}. \qquad (4.2)
$$

Thanks to (2.5), this potential is constant outside of some ball centered at the origin and satisfies the assumption of Theorem 3.1. We may thus apply this result with $U = V_B$ and the correlation factor F at hand. In view of (3.3) we have

$$
\mu^{(1)}_F(z) = (N-1)\rho_{\Psi_F}\left(\sqrt{N-1}\, z\right)
$$

and the theorem implies that there exists a ρ_F of unit L^1 norm satisfying

$$
0 \leq \rho_F \leq \frac{1}{\pi(\ell + m)} + \frac{\varepsilon \sup |\Delta V_B|}{\pi(\ell + m)}
$$

such that

$$\mathcal{E}_N[\Psi_F] = (N-1)\int_{\mathbb{R}^2} V_B(\mathbf{x})\rho_{\Psi_F}\left(\sqrt{N-1}\,\mathbf{x}\right)d\mathbf{x}$$
$$\geq \int_{\mathbb{R}^2} V_B\rho_F - C(N\varepsilon)^{-1}\mathrm{Err}(m,D,\varepsilon V_B). \tag{4.3}$$

Here, by assumption, m and D are fixed when $N\to\infty$. Passing then to the limit $N\to\infty$ at fixed ε we obtain

$$\liminf_{N\to\infty}\mathcal{E}_N[\Psi_F] \geq \inf\left\{\int_{\mathbb{R}^2} V_B\rho,\ 0\leq\rho\leq\frac{1}{\pi(\ell+m)}+\frac{\varepsilon\ \sup|\Delta V_B|}{\pi(\ell+m)}\right\}.$$

We may then pass to the limit $\varepsilon\to 0$:

$$\liminf_{N\to\infty}\mathcal{E}_N[\Psi_F] \geq \inf\left\{\int_{\mathbb{R}^2} V_B\rho,\ 0\leq\rho\leq\frac{1}{\pi(\ell+m)}\right\}$$

and finally to the limit $B\to\infty$, which yields

$$\liminf_{N\to\infty}\mathcal{E}_N[\Psi_F] \geq E_V\,(\ell+m)\geq E_V(\ell).$$

The necessary continuity of the bath-tub energy (2.6) as a function of the upper bound on the admissible trial states and the cut-off of the potential are easily deduced from the explicit formulae of, e.g., [LL, Theorem 1.14]. In fact, if B is large enough the bath-tub energy in the truncated potential V_B is equal to the bath-tub energy in the potential V. □

4.2. **Optimality in radial potentials: proof of Corollary 2.3.** Given Theorem 2.1, the only thing left to do is the proof of (2.11) and (2.13). We thus consider the special trial functions (2.12), that are built using correlation factors of the form

$$F(z_1,\ldots,z_N)=\prod_{j=1}^N z_j^m,\quad m\in\mathbb{N}. \tag{4.4}$$

The corresponding classical Hamiltonians (3.5) that we associate to them in order to analyze the one-body densities of the states Ψ_F have purely Coulomb two-body interactions ($g_2\equiv 1$ in this case). One can take advantage of this fact to analyze the classical mean-field limit with a different method than that we used in Section 3. This was done in [RSY2] and the method leads to somewhat stronger estimates.

We continue to use the notation of Section 3 to quote some results from [RSY2]. Thus $\mu_F^{(1)}$ is the rescaled one-body density of the state Ψ_F. Ultimately it will be sufficient to take $m\sim CN$ in the limit $N\to\infty$, so we may[9] invoke [RSY2, Theorem 3.1, Item 1]: for any regular enough $U:\mathbb{R}^2\mapsto\mathbb{R}$

$$\left|\int_{\mathbb{R}^2} U\left(\mu_F^{(1)}-\varrho^{\mathrm{MF}}\right)\right| \leq CN^{-1/2}\log N\|\nabla U\|_{L^2(\mathbb{R}^2)}+CN^{-1/2}\|\nabla U\|_{L^\infty(\mathbb{R}^2)} \tag{4.5}$$

where “regular enough” only means that we require the norms appearing on the right-hand side of the above equation to be finite. Here ϱ^{MF}, which was denoted ϱ^{el} in [RSY2], is the

[9]In [RSY2] we only considered the case $\ell=2$, but the whole analysis adapts to any fixed ℓ. Lengths were scaled by a factor $\sqrt{N}$ instead of $\sqrt{N-1}$, so the formulas we quote have to be slightly modified.

minimizer of the functional (3.10) corresponding to the choice (4.4). This means that we take $m = 0$, $g_2 \equiv 1$ and

$$-\frac{1}{N-1}\log|g_1(z)| = -\frac{m}{N-1}\log|z|.$$

In this case, $\varrho^{\rm MF}$ can be explicitly computed, see [RSY2, Proposition 3.1]:

$$\varrho^{\rm MF} = \frac{1}{\ell\pi}\mathbb{1}_{B(0,\sqrt{\ell})} \text{ if } m = 0 \tag{4.6}$$

$$\varrho^{\rm MF} = \frac{1}{\ell\pi}\mathbb{1}_{\mathcal{A}_N} \text{ if } m > 0 \tag{4.7}$$

where $\mathcal{A}_N$ is the annulus of inner radius $R_m^- = \sqrt{m/(N-1)}$ and outer radius $R_m^+ = \sqrt{(\ell+m)/(N-1)}$ centered at the origin.

Of course we cannot apply (4.5) directly with $U = V$, since the norms involved in the estimate are infinite for the latter potential. We thus first split V in two parts

$$V(\mathbf{x}) = \chi_{\rm in}(\mathbf{x})V(\mathbf{x}) + \chi_{\rm out}(\mathbf{x})V(\mathbf{x}) \tag{4.8}$$

using a smooth partition of unity $\chi_{\rm in} + \chi_{\rm out} \equiv 1$, where $\chi_{\rm in} = 1$ in $B(0,R)$ and $\chi_{\rm in} = 0$ in $B(0,2R)^c$ for some R to be chosen later on. We will use (4.5) to deal with the $\chi_{\rm in}V$ part, and show the contribution of the $\chi_{\rm out}V$ part to be negligible using

$$\mu_F^{(1)}(z) \leq C_1 \exp\left(-C_2 N\left(\left(|z| - \sqrt{\frac{m}{N-1}}\right)^2 - \log N\right)\right) \quad \text{when } \left||z| - \sqrt{\frac{m}{N-1}}\right| \geq C_3. \tag{4.9}$$

which is [RSY2, Equation (3.16)] and we recall that $m \sim CN$ in our case. This estimate implies that for any power $\alpha > 0$ and N large enough

$$\mu_F^{(1)}(z) \leq C_1 \exp\left(-C_2 N|z|^2\right) \text{ when } |z| \geq N^\alpha, \tag{4.10}$$

where the value of the constants C_1, C_2 have changed. Choosing $R = N^\alpha$ for some small (but fixed) power $\alpha > 0$ we then clearly have

$$(N-1)\int_{\mathbb{R}^2} \chi_{\rm out}(\mathbf{x})V(\mathbf{x})\rho_{\Psi_F}\left(\sqrt{N-1}\,\mathbf{x}\right)d\mathbf{x} = \int_{\mathbb{R}^2}\chi_{\rm out}(\mathbf{x})V(\mathbf{x})\mu_F^{(1)}(\mathbf{x})\,d\mathbf{x} \to 0 \tag{4.11}$$

when $N \to \infty$. This is the one place where we use the assumption that V grows at most polynomially at infinity (which, in view of (4.10) could be relaxed a bit).

Next we can use (4.5) to show that

$$\begin{aligned}(N-1)\int_{\mathbb{R}^2} \chi_{\rm in}(\mathbf{x})V(\mathbf{x})\rho_{\Psi_F}\left(\sqrt{N-1}\,\mathbf{x}\right)d\mathbf{x} &= \int_{\mathbb{R}^2}\chi_{\rm in}(\mathbf{x})V(\mathbf{x})\mu_F^{(1)}(\mathbf{x})\,d\mathbf{x} \\ &\sim \int_{\mathbb{R}^2} V(\mathbf{x})\varrho^{\rm MF}(\mathbf{x})d\mathbf{x} \text{ when } N\to\infty. \end{aligned}\tag{4.12}$$

We have already estimated very similar terms in [RSY2, Section 4] and will not reproduce all computations here. Simply observe that $\chi_{\rm in}$ can clearly be chosen with $|\nabla\chi_{\rm in}| \leq CN^\alpha$. Also, since we are free to choose the power α as small as desired, using the assumption that V grows at most poynomially at infinity, we may guarantee that the norms of $U = \chi_{\rm in}V$

that appear in the right-hand side of (4.5) grow at most as $N^{c\alpha}$ for some constant $c > 0$. This allows to control the error term as in [RSY2, Section 4], and proves (4.12).

Gathering (4.8), (4.11) and (4.12) we have now proved that

$$\mathcal{E}_N[\Psi_F] \sim \int_{\mathbb{R}^2} V(\mathbf{x})\varrho^{\mathrm{MF}}(\mathbf{x})d\mathbf{x} \text{ when } N \to \infty \tag{4.13}$$

where F is as in (4.4) and we have assumed $m \sim CN$. We now discuss this final result as a function of

$$\underline{m} = \lim_{N\to\infty} \frac{m}{N}. \tag{4.14}$$

We have, in L^∞ norm,

$$\varrho^{\mathrm{MF}} \to \frac{1}{\ell\pi}\mathbb{1}_{B(0,\sqrt{\ell})} \text{ if } \underline{m} = 0 \tag{4.15}$$

$$\varrho^{\mathrm{MF}} \to \frac{1}{\ell\pi}\mathbb{1}_{\sqrt{\underline{m}}\leq|\mathbf{x}|\leq\sqrt{\ell+\underline{m}}} \text{ if } \underline{m} > 0, \tag{4.16}$$

which allows to conclude the proof. Indeed, it is well-known [LL, Theorem 1.14] that the infimum in the bath-tub energy (2.6) is attained for a density saturating the constraint $\rho \leq (\pi\ell)^{-1}$. The minimizer is explicit as a function of the potential V, and in the cases we consider here it is easy to see that it is exactly equal to (4.15), provided a proper choice of $\underline{m}$ is made.

In case (1) of Corollary 2.3 we take the pure Laughlin state, i.e. $F \equiv 1$ and so $\underline{m} = 0$, and we obtain

$$\mathcal{E}_N[\Psi_F] \to \frac{1}{\ell\pi}\int_{B(0,\sqrt{\ell})} V(\mathbf{x})d\mathbf{x} \text{ when } N \to \infty$$

and the latter quantity is of course equal to $E_V(\ell)$, the bath-tub energy defined in (2.6). Indeed, if V is radial increasing, the minimizer of the bath-tub energy is simply (4.15): the density has to saturate the bound $\rho \leq (\pi\ell)^{-1}$ on its support, and it is clear that the optimal choice is to take this support to be a disc centered on the minimum of V, i.e at the origin. This proves (2.11).

In case (2), a similar reasoning yields that the minimizer of $E_V(\ell)$ is given by

$$\frac{1}{\ell\pi}\mathbb{1}_{A\leq r\leq B}$$

for some $A, B > 0$ tuned so that the above function is normalized in L^1, i.e. $B^2 - A^2 = \ell$. Choosing $\underline{m}$ so that

$$A = \sqrt{\underline{m}} \text{ and } B = \sqrt{\ell + \underline{m}},$$

that is, taking m to be the integer part of NA^2 in (2.12) we deduce that also in this case

$$\mathcal{E}_N[\Psi_F] \to E_V(\ell) \text{ when } N \to \infty,$$

which is (2.13).

If D is large enough, the trial states we have just built indeed all belong to $\mathcal{V}_2^D$, so we deduce from (2.11)-(2.13) that

$$E_2^D(N) \leq \mathcal{E}_N[\Psi_F] \to E_V(\ell)$$

which combines with (2.7) to complete the proof of (2.10).

□

5. Extensions of the main results

Energy lower bounds of the type (2.7) are a manifestation of the incompressibility of the states involved and ideally one would like to derive them for all fully correlated states (1.4), not only the special states considered in Theorem 2.1. This is a quite ambitious goal and genuinely new ideas will be needed to achieve it completely. However, substantial generalizations of Theorem 2.1 can be achieved by our methods as we now discuss.

The main generalization one could handle with our methods corresponds to allow "n-body correlation factors" of the form

$$F(z_1,\ldots,z_N)=\prod_{j=1}^N f_1(z_j)\prod_{(i,j)\in\{1,\ldots,N\}} f_2(z_i,z_j)\ldots\prod_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}} f_n(z_{i_1},\ldots,z_{i_n}) \quad (5.1)$$

for some finite fixed n and symmetric holomorphic functions $f_1,\ldots,f_n\in\mathcal{B}^1,\ldots,\mathcal{B}^n$ that can depend on N, with no a priori bounds on their degree. Let us sketch briefly these possible improvements:

Removing a priori bounds on the degree. In (2.1) we have restricted our attention to polynomials f_1 and f_2 satisfying a priori bounds on their degrees. First note that the estimates of Theorem 3.1 are actually explicit as a function of this degree so the theorem is still valid if the assumption is relaxed, with worse remainder terms however.

In fact we claim that if either bound is violated for a sequence

$$F(z_1,\ldots,z_N)=\prod_{j=1}^N f_1(z_j)\prod_{(i,j)\in\{1,\ldots,N\}} f_2(z_i,z_j)$$

then the rescaled density defined by (3.3) satisfies

$$\mu_F^{(1)}\rightharpoonup 0 \quad (5.2)$$

weakly as measures, which clearly implies

$$\mathcal{E}_N[\Psi_F]\to+\infty \text{ when } N\to\infty$$

because of (2.5) and justifies the a priori reduction we made.

More precisely in this case one has

$$\mu_F^{(1)}(B(0,R))\to 0 \text{ when } N\to\infty \quad (5.3)$$

for any fixed radius R. The computations leading to (5.3) are a bit tedious, especially for general polynomials f_1 and f_2, but here is the main idea: If either g_1 or g_2 has a large degree in the classical Hamiltonian (3.5), then it is clear that the two-body potential

$$\begin{aligned}\bar W(z,z'):=\frac{|z|^2}{2}+\frac{|z'|^2}{2}-\frac{2}{N-1}\log|g_1(z)|-\frac{2}{N-1}\log|g_1(z)|\\ -2(\ell+m)\log|z-z'|-2\log|g_2(z,z')| \quad (5.4)\end{aligned}$$

takes its minimum far from the origin, in fact infinitely far in the limit $N\to\infty$. Simple model cases are e.g. $f_1\equiv 1$, $f_2\equiv 1$ and $m\to\infty$ when $N\to\infty$, or[10] $f_1(z)=z^\gamma$, $f_2\equiv 1$

[10] This latter case has been considered in [RSY2].

and $m = 0$ with $\gamma \gg N$. Since the free-energy of μ_F in the plasma analogy is

$$N \iint_{\mathbb{R}^2\times\mathbb{R}^2} \bar{W}(z,z')\mu_F^{(2)}(z,z')dzdz' + \frac{1}{N-1}\int_{\mathbb{R}^{2N}} \mu_F \log \mu_F$$

it is clear that it will be favorable for μ_F to have its mass concentrated far from the origin. One can easily construct factorized trial states having this behavior and compare their free-energy to that of the minimizer μ_F, taking into account that $\bar{W}$ is much larger than its infimum in any ball $B(0,R)$ with R fixed. Rather simple arguments then allow to deduce (5.3) in good cases, but the computations allowing to control (5.4) for generic polynomials f_1, f_2 are rather tedious. Arguments of this sort could also be adapted to rule out more general holomorphic functions than polynomials.

Higher order correlation factors. Recall the definition (1.5) of the N-body bosonic Bargmann space and define the set

$$\mathcal{V}_n = \Big\{ F \in \mathcal{B}^N, \text{ there exist } (f_1,\ldots,f_n) \in \mathcal{B}\times\ldots\times\mathcal{B}^n,$$
$$F(z_1,\ldots,z_N) = \prod_{j=1}^N f_1(z_j) \prod_{(i,j)\in\{1,\ldots,N\}} f_2(z_i,z_j)\ldots \prod_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}} f_n(z_{i_1},\ldots,z_{i_n}) \Big\} \tag{5.5}$$

where $(i_1,\ldots,i_n)$ is understood as an n-tuple with no repetition of any of the indices $i_k, k\in\{1,\ldots,n\}$. We have

- $\mathcal{V}_n \subset \mathcal{V}_{n+1}$: given $F_n \in \mathcal{V}_n$ associated to $(f_1,\ldots,f_n)\in\mathcal{B}\times\ldots\times\mathcal{B}^n$ we can see it as an element of $\mathcal{V}_{n+1}$ associated with $(f_1,\ldots,f_n,1)\in\mathcal{B}\times\ldots\times\mathcal{B}^n\times\mathcal{B}^{n+1}$.
- $\mathcal{V}_N = \mathcal{B}^N$: for $F_N \in \mathcal{B}^N$ one may simply choose $(1,1,\ldots,F_N)\in\mathcal{B}\times\ldots\times\mathcal{B}^N$.

We can consider the minimization of (1.7) amongst states with the correlation factors (5.5) instead of the simpler (2.1). This way we obtain the family of energies

$$E_n(N) := \inf\{\mathcal{E}_N[\Psi_F],\ \Psi_F \text{ of the form (2.2) with } F\in\mathcal{V}_n\}. \tag{5.6}$$

and since $\mathcal{V}_n \subset \mathcal{V}_{n+1}$ we have of course

$$E(N) = E_N(N) \leq \ldots \leq E_{n+1}(N) \leq E_n(N) \leq \ldots \leq E_N(1). \tag{5.7}$$

One may expect that equality holds, at least asymptotically for large N, in a large number of these equalities. A first step would be to generalize Theorem 2.1 to a lower bound on $E_n(N)$ for n as large as possible.

In case $F\in\mathcal{V}_n\setminus\mathcal{V}_{n-1}$ for some fixed finite $n\geq 3$ one should in fact also expect

$$\mu_F^{(1)} \rightharpoonup 0, \tag{5.8}$$

with the same consequences as before. Indeed, the plasma analogy also applies to functions built on $\mathcal{V}_n$, leading now to a classical energy including n-body terms, but now because of the symmetry there are at least $\binom{N}{n}$ interaction terms. Because of the scaling in (3.3), these come multiplied by a prefactor $\propto N^{-1}$, so that the total interaction strength will be of order roughly $N^{-1}\binom{N}{n} \gg 1$ if $n>2$. With such a huge strength, it is intuitive that (5.8) should occur. In fact, to see a non trivial behavior of $|\Psi_F|^2$, one should rather rescale it on a much larger length-scale. To make this more precise one can inspect the resulting n-body potential $\bar{W}_n$ replacing (5.4) and see that its minimum would occur again very far from the origin. If $n\geq 3$, this will happen even if the n-body correlation factors have bounded degree. Since the classical free-energy of the classical plasma will now have the form

$$N \int_{\mathbb{R}^{2n}} \bar{W}_n(z_1, \dots, z_n) \mu_F^{(n)}(z_1, \dots, z_n) dz_1, \dots, dz_n + \frac{1}{N-1} \int_{\mathbb{R}^{2N}} \mu_F \log \mu_F$$

one can argue as before that (5.8) occurs.

Of course these (formal) arguments break down if only the correlation factors f_n, $n \geq N-1$ are non trivial. In this case the total interaction strength is again of order $N^{-1}\binom{N}{n} \lesssim 1$, so even if these ideas could be made rigorous, they would not allow to deal with the full set $\mathrm{Ker}(\mathcal{I}_N)$. One could still investigate for example the case of n fixed when $N \to \infty$ and deduce that in this case

$$\liminf_{N \to \infty} E_n(N) \geq E_V(\ell).$$

Combining with the arguments of Corollary 2.3, one would then obtain

$$\lim_{N \to \infty} E_n(N) = \lim_{N \to \infty} E_m(N)$$

for any fixed $n, m \in \mathbb{N}$ in the case of radial increasing or radial mexican-hat potentials.

Concerning the analysis of the states (5.1) we finally note that one could also extract some useful information from the mean-field approximation procedure of Section 3. It would be useful to scale space variables differently but this is a detail. One would need to work with the formulas (A.3) for higher-order marginals in the Diaconis-Freedman theorem. In view of (A.2), it should be possible to obtain quantitative information as long as $n \ll \sqrt{N}$.

Appendix A. The Diaconis-Freedman theorem

We make use of the Diaconis-Freedman theorem [DF], which may be seen as a quantitative version of the Hewitt-Savage (or classical de Finetti) theorem [HS], whose importance for classical mean-field problems has been recognized for some time now [CLMP, Kie, KS, MS, Gol]. The Hewitt-Savage theorem is most often seen as an existence result in the literature, but it in fact follows from the constructive approach in [DF]. Since we make use of this fact, it is worth recalling the main result of [DF] and sketching the proof. We shall denote $\| \, . \, \|_{\mathrm{TV}}$ the total variation norm.

Theorem A.1 (Diaconis-Freedman).
Let S be a measurable space and $\mu \in \mathcal{P}_s(S^N)$ be a probability measure on S^N invariant under permutation of its arguments. There exists $P_\mu \in \mathcal{P}(\mathcal{P}(S))$ a probability measure such that, denoting

$$\tilde{\mu} := \int_{\rho \in \mathcal{P}(S)} \rho^{\otimes N} dP_\mu(\rho) \tag{A.1}$$

we have

$$\left\| \mu^{(n)} - \tilde{\mu}^{(n)} \right\|_{\mathrm{TV}} \leq \frac{n(n-1)}{N}. \tag{A.2}$$

In addition, the marginals of $\tilde{\mu}$ are given by :

$$\tilde{\mu}^{(n)}(x_1, \dots, x_n) = \frac{1}{N^n} \sum_{j=1}^{n} \frac{N\,!(n-j)\,!}{(N-j)\,!\,n\,!} \sum_{\sigma \in \Sigma_n} \mu^{(j)}(x_{\sigma(1)}, \dots, x_{\sigma(j)})\, \delta_{x_{\sigma(j)} = x_{\sigma(j+1)} = \dots = x_{\sigma(n)}} \tag{A.3}$$

with Σ_n the group of permutations of n elements.

Proof. The proof may be found in [DF], we provide a sketch for the convenience of the reader. We take $S = \mathbb{R}^2$, which is the case of interest for us here and abuse notation by writing $\mu(Z)dZ$ instead of $d\mu(Z)$ for integrals in $Z \in \mathbb{R}^{2N}$. Note that the symmetry of μ implies

$$\mu(X) = \int_{\mathbb{R}^{2N}} \mu(Z) \sum_{\sigma \in \Sigma_N} (N!)^{-1} \delta_{Z_\sigma = X} dZ. \tag{A.4}$$

The main idea is to define

$$\tilde{\mu}(X) = \int_{\mathbb{R}^{2N}} \mu(Z) \sum_{\gamma \in \Gamma_N} N^{-N} \delta_{Z_\gamma = X} dZ, \tag{A.5}$$

with Γ_N the set of all maps (not necessarily one-to-one) from $\{1, \ldots, N\}$ to itself. Noticing that

$$\sum_{\gamma \in \Gamma_N} N^{-N} \delta_{Z_\gamma = X} = \left(N^{-1} \sum_{j=1}^{N} \delta_{z_j = x} \right)^{\otimes N}, \tag{A.6}$$

one may put (A.5) in the form (A.1) by taking

$$P_\mu(\rho) = \int_{\mathbb{R}^{2N}} \delta_{\rho = \bar{\rho}_Z} \mu(Z) dZ, \quad \bar{\rho}_Z(x) := \sum_{i=1}^{N} N^{-1} \delta_{z_j = x}. \tag{A.7}$$

Computing the difference between $\mu^{(n)}$ and $\tilde{\mu}^{(n)}$ we have of course

$$\mu^{(n)} - \tilde{\mu}^{(n)} = \int_{\mathbb{R}^{2N}} \left(\left(\sum_{\sigma \in \Sigma_N} (N!)^{-1} \delta_{Z_\sigma = X} \right)^{(n)} - \left(\sum_{\gamma \in \Gamma_N} N^{-N} \delta_{Z_\gamma = X} \right)^{(n)} \right) \mu(Z) dZ$$

but $\sum_{\sigma \in \Sigma_N} (N!)^{-1} \delta_{Z_\sigma = X}$ is the probability law of drawing N balls at random from an urn [11], *without* replacement whereas $\sum_{\gamma \in \Gamma_N} N^{-N} \delta_{Z_\gamma = X}$ is the probability law of drawing N balls at random from an urn, *with* replacement. It is thus intuitively clear that the difference

$$\left(\sum_{\sigma \in \Sigma_N} (N!)^{-1} \delta_{Z_\sigma = X} \right)^{(n)} - \left(\sum_{\gamma \in \Gamma_N} N^{-N} \delta_{Z_\gamma = X} \right)^{(n)}$$

between their reduced densities is small when $n \ll N$. The meaning of 'small' in this sentence is not difficult to quantify as a function of n and N, see [Fre] where the total variation bound $\frac{n(n-1)}{N}$ is obtained, which leads to (A.2).

[11] Where the balls are labeled $x_1, \ldots, x_N$.

The only fact that is not explicitly mentioned in [DF] is (A.3), but this is an easy consequence of (A.6). Using the symmetry of P we have

$$\tilde{\mu}^{(1)}(x) = N^{-1} \sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} \mu(Z)\delta_{z_j=x} dZ = \mu^{(1)}(x)$$

$$\begin{aligned}
\tilde{\mu}^{(2)}(x_1,x_2) &= N^{-2} \int_{\mathbb{R}^{2N}} \mu(Z) \left(\sum_{j=1}^{N} \delta_{z_j=x_1} \right) \left(\sum_{j=1}^{N} \delta_{z_j=x_2} \right) dZ \\
&= N^{-2} \sum_{1\leq i\neq j\leq N} \int_{\mathbb{R}^{2N}} \mu(Z)\delta_{z_i=x_1}\delta_{z_j=x_2} dZ + N^{-2} \sum_{i=1}^{N} \int_{\mathbb{R}^{2N}} \mu(Z)\delta_{z_i=x_1}\delta_{z_i=x_2} dZ \\
&= \frac{N-1}{N} \mu^{(2)}(x_1,x_2) + \frac{1}{N}\mu^{(1)}(x_1)\delta_{x_1=x_2}. \qquad \text{(A.8)}
\end{aligned}$$

The computation of the higher order marginals follows along the same lines and leads to (A.3). An estimate of the form (A.2) can also be seen as following from this computation as noted by Lions [Lio].

□

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